

# Renormalization group and continuum limit in Quantum Mechanics

Janos Polonyi<sup>a</sup>

<sup>a</sup>Department of Physics, University Louis Pasteur,  
Strasbourg, France and Department of Atomic Physics, L. Eötvös University,  
Budapest, Hungary

The running coupling constants are introduced in Quantum Mechanics and their evolution is described by the help of the renormalization group equation.

## 1. INTRODUCTION

The typical trajectories of the path integral for a free nonrelativistic particle are nowhere differentiable [1]. In fact, the dominant contributions to the amplitude

$$\prod_k \int dx_k e^{\frac{im}{2\hbar\Delta t} \sum_k \Delta x_k^2}, \quad (1)$$

$\Delta x_k = x_{k+1} - x_k$ , is such that the contributions to the sum in the exponent is order one, i.e.  $\frac{\Delta x}{\Delta t} = O(\sqrt{\frac{1}{\Delta t}})$ . In other words, the determination of the velocity within the time span  $\Delta t$  yields fluctuations  $O(\sqrt{\frac{1}{\Delta t}})$ . The intrinsic ‘disorder’ which might be interpreted as an evidence of the fractal structure of the trajectories represents the core of Quantum Physics for the canonical commutation relations would be lost with  $\frac{\Delta x}{\Delta t} = o(\sqrt{\frac{1}{\Delta t}})$ . Furthermore, this non-differentiability of the trajectories is the source of the Itô calculus [2] in Quantum Mechanics. One expects that the velocity dependent interactions become more important for short time processes. Our goal is to see whether the concept of running coupling constant is meaningful in Quantum Mechanics and it shows an enhancement for velocity dependent interactions at high energy or short time.

Quantum Mechanics can formally be considered as a lattice regulated Quantum Field Theory in 0+1 dimension with lattice spacing  $\Delta t$ . One finds that the transition amplitudes are not necessarily ultraviolet finite when velocity dependent interactions are present. The power counting ar-

gument shows that the vertex  $(\frac{\Delta x}{\Delta t})^d x^s$  is renormalizable, i.e. its contributions do not become more and more ultraviolet singular in the higher orders of the perturbation expansion for  $d \leq 2 + s$  [3]. We shall find that some of the ultraviolet singularities predicted by the simple power counting argument are curiously enough cancel in the simple perturbation expansion and are only present when the expansion is made around non-static trajectories. This raises the question about the equivalence of the operator and the path integral formalism. These questions will be considered here in the framework of the perturbation expansion.

The renormalization of Quantum Mechanics has already been discussed in the presence of singular potential in [4]. Our motivation to look for the running coupling constant is different, the goal is now to understand the time dependence of the transition amplitude

$$\langle x | e^{-\frac{i}{\hbar} t H} | y \rangle = \langle x | U(t) | y \rangle = e^{\frac{i}{\hbar} S(x, y; t)} \quad (2)$$

for general regular interaction [3].

## 2. RUNNING COUPLING CONSTANTS

The running coupling constants,  $g_{ds}(t)$ , are defined in the spirit of the Landau-Ginsburg expansion by

$$S(x, y; t) = \sum_{ds} g_{ds}(t) (x - y)^d (x + y)^s, \quad (3)$$

and they characterize the transition probabilities. The vector indices are suppressed here for the sake of simplicity. The coupling constants of the

lagrangian is  $g_{ds} = t^{-2}\tilde{g}_{ds}$  after the separation of the time dimension due to the cut-off  $\Lambda = 2\pi/\Delta t$  and  $\hbar$ .

The behavior of these effective coupling constants is rather nontrivial even for a harmonic oscillator. The running mass and frequency, given by

$$m(t) = 2tg_{20}(t) = \frac{tm(0)\omega_0(1 + \cos\omega(0)t)}{\sin\omega(0)t} \quad (4)$$

and

$$\omega^2(t) = \frac{8g_{02}(t)}{tm(t)} = \frac{4}{t^2} \frac{1 - \cos\omega(0)t}{1 + \cos\omega(0)t} \quad (5)$$

show periodicity in real time, a characteristic feature for equidistant energy spectrum.

### 3. RENORMALIZATION GROUP EQUATION

The renormalization group equation is given by

$$e^{\frac{i}{\hbar}S(x,y;t_1+t_2)} = \int dz e^{\frac{i}{\hbar}S(x,z;t_1)} e^{\frac{i}{\hbar}S(z,y;t_2)}. \quad (6)$$

The decimation consists of choosing  $t_1 = t_2 = t$  and the linearization of the resulting blocking transformation  $t \rightarrow t' = 2t$  yields the tree level scaling law

$$\tilde{g}_{ds}(t') = \left(\frac{t'}{t}\right)^{2-d} \tilde{g}_{ds}(t). \quad (7)$$

Thus the relevant or marginal pieces of the lagrangian are the kinetic energy, the vector and the scalar potential. Note that the vertices with  $2 < d < 2 + s$  are renormalizable and irrelevant in the same time. This results from the loss of the relativistic invariance, i.e. the difference of the energy and the inverse time dimensions. In fact, the former controls the ultraviolet divergences while the latter is responsible for the scaling exponents when the blocking is made in the time direction.

The one-loop evaluation of the differential renormalization group equation in the limit  $t_2 \rightarrow 0$  gives the Schrödinger equation for the logarithm of the wave function. This method establishes a transparent relation between the transition amplitude,  $e^{-\frac{i}{\hbar}S(x,y;t)}$ , and the hamiltonian,  $H(t) = -\frac{\hbar}{it} \ln U(t)$ .

### 4. OPERATOR ORDERING AND RENORMALIZATION

Eq. (7) gives the cut-off independence of the mass for  $d = 2$  and  $s = 0$ . Another less trivial implication of this equation is the requirement of the midpoint prescription for the vector potential. The lagrangian

$$\frac{m}{2} \left( \frac{\Delta x_k}{\Delta t} \right)^2 + \frac{\Delta x_k}{\Delta t} A \left( \frac{x_{k+1} + x_k}{2} + \eta \Delta x_k \right), \quad (8)$$

generates gauge dependent amplitudes unless  $\eta = 0$  [5]. This quantum effect results from the marginality of the term  $\Delta x^2$  which gives the leading order  $\eta$  dependence of the path integral.

Since the saddle point of the integral in the renormalization group is different for  $t_1 = t_2$  and  $t_1 \neq t_2 \rightarrow 0$  the one-loop scaling relations might well be different for the lagrangian and the hamiltonian. It turns out that the contribution to the renormalization group equations which is responsible for the midpoint prescription is logarithmically divergent in the blocking but remain finite in the framework of the simple perturbation expansion or in the hamiltonian. In fact, the leading order  $\eta$  dependence of the path integral is proportional to  $\mathcal{M} = \Delta t \sum_k < \left( \frac{\Delta x_k}{\Delta t} \right)^2 >_0$ , where  $< \dots >_0$  stands for the free expectation value. Thus  $\mathcal{M} = I\Lambda^{-1}$ , where  $I$  is a linearly divergent loop integral,  $I = A\Lambda + B\Lambda \ln \Lambda$ . In the expansion around a static trajectory or in the derivation of the hamiltonian the logarithmic correction is absent. Thus the product  $\mathcal{M} = \frac{1}{\Lambda} A\Lambda$  is finite, in a manner which is reminiscent of the chiral anomaly. The perturbation expansion in the decimation is performed around a general, non-static trajectory with time scale. This scale lets the usual logarithmic corrections to appear. In other words, a graph with zero primitive degree of divergence happened to be finite in the expansion around the static trajectory. The hamilton formalism based on such expansion is well defined without the introduction of the ultraviolet cut-off. When the background trajectory is chosen to be non-static the logarithmic divergence appears and makes regularization necessary for the path integral.

The presence of the potentially divergent but actually finite graph is reflected in the operator ordering problem in Quantum Mechanics. The  $\eta$  dependent vertices which are vanishing in the classical limit i.e. for differentiable trajectories survive the removal of the cut-off. This is in an apparent contradiction with the result of Ref. [6] where the renormalization of the lattice regulated field theories was studied in the BPHZ scheme. The finiteness of  $\mathcal{M}$  in the framework of the simple perturbation expansion makes renormalization unnecessary but leaves a non-uniformly convergent loop integral in the theory. In the BPHZ program all graphs with nonpositive primitive degree of divergence are subtracted and the convergence of the manifestly convergent loop integrals is uniform. It remains to be seen whether the operator ordering problem can change the renormalization properties in Quantum Field Theory.

## 5. EFFECTIVE LOW ENERGY THEORIES

The renormalization group flow makes possible to construct low energy effective models in Quantum Mechanics. According to the leading order perturbation expansion the action  $S(x, y; t) = \frac{m(x-y)^2}{2t} - tU(x, y; t)$  where the potential  $U(x, y; t)$  given by either

$$V\left(\frac{x+y}{2}\right), \quad (9)$$

or

$$\frac{1}{(d/2)!} \left(\frac{m}{3i\hbar}\right)^{d/2} \left(\frac{x-y}{\sqrt{t}}\right)^d V\left(\frac{x+y}{2}\right), \quad (10)$$

generates the same hamiltonian,

$$H = \frac{p^2}{2m} + V(x), \quad (11)$$

for small but finite  $t$ . The potential (10) does not follow the renormalized trajectory in the limit  $t \rightarrow 0$  and the perturbation expansion breaks down for small enough  $t$ . Thus the two potential contain the same relevant terms in the hamiltonian continuum limit, (11).

## 6. PATH INTEGRAL VERSUS OPERATOR FORMALISM

The low energy effective models of Quantum Field Theory may contain non-renormalizable operators. In this manner the class of effective theories is larger than those of the fundamental, i.e. renormalizable theories. For example the hadronic effective vertices which are described by multi-quark operators are certainly generated in low energy QCD but can not be explicitly present in the renormalizable QCD. Thus certain physically well motivated vertices of the low energy theory can only be kept meaningful in cut-off theories. As we insist on the removal of the cut-off then these irrelevant operators must be eliminated from the bare lagrangian and they can only be generated dynamically.

The effective model mentioned above shows the same features. The scalar potential in (9) is well defined for arbitrary values of the cut-off. The system (10) contains additional irrelevant terms which can not be obtained directly from a hamiltonian which is given for  $\Delta t = 0$ . In this sense the cut-off version of Quantum Mechanics which is given by the path integral in terms of the transition amplitudes for infinitesimal time is more general than the operator formalism where the equation of motion is a differential equation with the cut-off already removed.

## REFERENCES

1. R. P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integrals*, McGraw-Hill, New York, 1965.
2. K. Itô, Mem. Am. Math. Soc. No. 4 (1951).
3. J. Polonyi, "Renormalization Group in Quantum Mechanics", submitted to Nucl. Phys., hep-th/9409004.
4. K.S. Gupta and S.G. Rajeev, Phys. Rev. D48 (1993) 5940, hep-th/9305052; C. Manuel and R. Tarrach, Phys. Lett. B328 (1994) 113, hep-th/9309013; P. Gosdinsky and R. Tarrach, Amer. J. Phys.59 (1994) 70.
5. L.S. Schulman, *Techniques and Applications of Path Integrations*, Wiley, New York, 1981.
6. T. Riesz, Comm. Math. Phys. **117** 639.